Repeated games and convergence

Convergence

In repeated games, it is often useful to know how geometric series converge. Here are some useful convergences. We assume that $1 < \delta < 0$ and that K is a real number.

1. $$K + K\delta + K\delta^2 + K\delta^3 + ... = \frac{K}{1-\delta}$$

2. $$K\delta + K\delta^2 + K\delta^3 + ... = \frac{K\delta}{1-\delta}$$

3. $$K + K\delta^2 + K\delta^4 + K\delta^6 + ... = \frac{K}{1-\delta^2}$$

4. $$K\delta^2 + K\delta^4 + K\delta^6 + ... = \frac{K\delta^2}{1-\delta^2}$$

5. These results can help us solve cases like the following:

$$K\delta^3 + K\delta^5 + K\delta^7 + ... = K\delta(\delta^2 + \delta^4 + \delta^6) = \frac{K\delta^3}{1-\delta^2}$$

General tips

- Remember that there are many ways to deviate, but in most cases only 1 or 2 make sense to be analyzed (for example, in trigger strategies, the optimal moment to deviate is to deviate in the first stage of the game and that is why only that type of deviation is analyzed).
- When you are not sure how a geometric series converges, the ideal is to try to bring it to something known, for example, the fourth case mentioned above can be solved with the previous cases once a common factor is taken out.
- The typical prisoner's dilemma always has the same solution, knowing the solution in advance can help to check when dealing with numerical values instead of parameters. (See below how it is solved generically with the trigger strategy).

Trigger strategy

Here is the minimum δ for players to cooperate using trigger strategies. That is, players will cooperate until the other player decides to deviate, and once the player deviates, the other player will punish in all subsequent periods. In this game, the Nash equilibrium is (Y, B). But the Pareto efficient outcome is (X, A), so it constitutes an example of the prisoner's dilemma where: $\gamma < \theta$, $\kappa < \gamma$, $\alpha < \kappa < \gamma$. Cooperating would be maintaining (X, A), and on the other hand, deviating would be playing Y or B depending on the player.

		Player 2	
		A	B
	X	$\gamma;\gamma$	$\alpha;\theta$
Player 1	Y	$\theta;\alpha$	$\kappa;\kappa$

We solve for player 1, let's see that if they cooperate, the payment would be:

$$\gamma + \gamma\delta + \gamma\delta^2 + ... = \frac{\gamma}{1-\delta}$$

On the other hand, if they choose not to cooperate, they will be punished after the first stage (and forever). Therefore, the payment would be:

$$\theta + \kappa\delta + \kappa\delta^2 + ... = \theta + \frac{\kappa\delta}{1-\delta}$$

We look for the discount factor with which there are incentives to cooperate

$$\frac{\gamma}{1-\delta} \geq \theta + \frac{\kappa\delta}{1-\delta}$$

$$\gamma \geq \theta(1-\delta) + \kappa\delta$$

$$\gamma - \theta \geq -\theta\delta + \kappa\delta$$

$$\delta \geq \frac{\theta - \gamma}{\theta - \kappa}$$